4540. *Proposed by Prithwijit De.*

Given a prime p and an odd natural number k, do there exist infinitely many natural numbers n such that p divides $n^k + k^n$? Justify your answer.

We received 26 submissions of which 25 were correct and complete. We present the solution by the Missouri State University Problem Solving Group, slightly modified.

The answer is positive. We show the existence by constructing n explicitly. There are the following two cases.

- If $p \mid k$, then one can take n to be any multiple of p.
- If $p \nmid k$, then one can take any positive integer

  $$n \equiv p - 1 \pmod{p(p-1)}.$$

  Since $n \equiv -1 \pmod{p}$ and k is odd, we have $n^k \equiv (-1)^k \equiv -1 \pmod{p}$. Since $n \equiv 0 \pmod{p-1}$ and $p \nmid k$, by Fermat's little theorem, $k^n \equiv 1 \pmod{p}$. Hence, $n^k + k^n \equiv -1 + 1 \equiv 0 \pmod{p}$.

Editor's Comment. As pointed out by UCLan Cyprus Problem Solving Group, the condition that k is odd is necessary. For example, if $k = 4$ and $p = 3$, then $n^4 + 4^n$ is never a multiple of 3 since $n^4 + 4^n \equiv 2 \pmod{3}$ when $3 \nmid n$ and $n^4 + 4^n \equiv 1 \pmod{3}$ when $3 \mid n$.