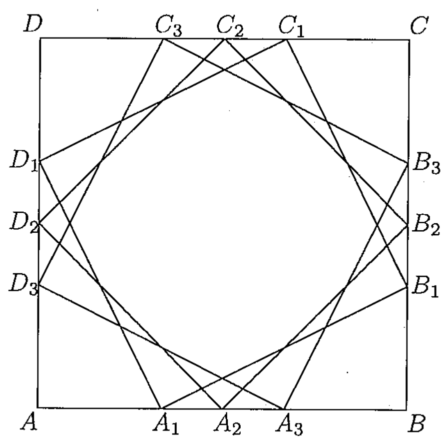
4638. *Proposed by Marie-Nicole Gras.*

We consider a square $ABCD$ of side $AB = 6a$, $a \in \mathbb{R}$; we put on the side AB points A_1, A_2, A_3 such that $AA_1 = 2a$, $A_1A_2 = A_2A_3 = a$, then we draw the squares $A_1B_1C_1D_1$, $A_2B_2C_2D_2$ and $A_3B_3C_3D_3$ as shown on the figure.



The region common to the interiors of the three squares is a dodecagon. Find the relationship between the areas of the dodecagon and the largest square.

This problem was inspired by 4449.

We received 15 submissions, all are correct. We present two solutions.

Solution 1, by Missouri State University Problem Solving Group, which contains a generalisation.

More generally, we will consider the case when

$$A = (-1,-1), B = (1,-1), C = (1,1), D = (-1,1),$$

and $A_1A_2 = A_2A_3 = \lambda, 0 < \lambda < 1$. Denote the intersection of $\overline{C_1D_1}$ and $\overline{C_3B_1}$ by P, the intersection of $\overline{C_2B_2}$ and $\overline{C_3B_3}$ by Q, and the intersection of $\overline{C_1B_1}$ and $\overline{C_2B_2}$ by R. Let O denote the origin. The dodecagon we are investigating consists of eight triangles congruent to $\triangle OPQ$ and four triangles congruent to $\triangle OQR$.

The equation of the line through C_1 and D_1 is

$$y = \frac{1-\lambda}{1+\lambda}x + \frac{1+\lambda^2}{1+\lambda}.$$

The point P corresponds to the y-intercept of this line, so

$$P = \left(0, \frac{1+\lambda^2}{1+\lambda}\right).$$

The equation of the line through B_3 and C_3 is

$$y = -\frac{1-\lambda}{1+\lambda}x + \frac{1+\lambda^2}{1+\lambda}$$

and the equation of the line through B_2 and C_2 is $y = 1 - x$. Finding the intersection of these two lines gives

$$Q = \left(\frac{1-\lambda}{2}, \frac{1+\lambda}{2}\right).$$

By symmetry, we have

$$R = \left(\frac{1+\lambda}{2}, \frac{1-\lambda}{2}\right).$$

The area of $\triangle OPQ$ is

$$\frac{1}{2}\det\begin{bmatrix}(1-\lambda)/2 & (1+\lambda)/2 \\ 0 & (1+\lambda^2)/(1+\lambda)\end{bmatrix} = \frac{(1-\lambda)(1+\lambda^2)}{4(1+\lambda)}.$$

The area of $\triangle OQR$ is

$$\frac{1}{2}\det\begin{bmatrix}(1+\lambda)/2 & (1-\lambda)/2 \\ (1-\lambda)/2 & (1+\lambda)/2\end{bmatrix} = \frac{\lambda}{2}.$$

Therefore the area of the dodecagon is

$$8\cdot\frac{(1-\lambda)(1+\lambda^2)}{4(1+\lambda)} + 4\cdot\frac{\lambda}{2} = \frac{2(1+2\lambda^2-\lambda^3)}{1+\lambda}.$$

The area of the larger square is 4, so the ratio of the area of the dodecagon to that of the square is

$$\frac{1+2\lambda^2-\lambda^3}{2(1+\lambda)}.$$

In the problem posed, $\lambda = 1/3$ and we obtain a ratio of $4/9$.

We note in passing that for a fixed larger square, the dodecagon's area is smallest when

$$\lambda = \frac{\sqrt{17}-3}{4}.$$