MA145. Determine all integers n for which $n^3 - 3n + 2$ is divisible by $2n + 1$.

Originally from 2010 Alberta High School Mathematics Competition, Part I, problem 16.

We received 14 submissions of which 9 were correct and complete. We present the solution by the Missouri State University Problem Solving Group.

Suppose n is an integer for which $n^3 - 3n + 2$ is divisible by $2n + 1$. By the division algorithm,

$$M = \frac{n^3 - 3n + 2}{2n + 1} = \frac{1}{2}n^2 - \frac{1}{4}n - \frac{11}{8} + \frac{27}{8(2n + 1)},$$

and M is an integer. Since

$$8M = 4n^2 - 2n - 11 + \frac{27}{2n + 1}$$

is also an integer, $2n + 1$ must be a divisor of 27. Solving $2n + 1 = d$ for each d in $\{\pm 1, \pm 3, \pm 9, \pm 27\}$, the set of divisors of 27, we get that n must be -14, -5, -2, -1, 0, 1, 4, or 13, and one readily verifies that these satisfy the condition.