OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2021: 47(3), p. 135–136.

OC521. In the plane there are two identical circles with radius 1, which are tangent externally. Consider a rectangle containing both circles, each side of which touches at least one of them. Determine the largest and the smallest possible area of such a rectangle.

Originally from 2018 Czech-Slovakia Math Olympiad, 3rd Problem, Category A, First Round.

We received 8 *solutions. We present* 2 *solutions.*

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Solution 2, by the Missouri State University Problem Solving Group.

We will assume that the bounding rectangle is oriented so that its sides are parallel to the coordinate axes and its center is at the origin. Suppose the two circles are tangent at the origin, arranged as follows. One circle has center $(\cos t, \sin t)$, and the other circle has center $(-\cos t, -\sin t)$, for $0 \le t \le \pi/2$. The bounding rectangle has dimensions $2(1+\cos t)$ and $2(1+\sin t)$. We wish to find the largest and smallest values of the area function $A(t) = 4(1+\cos t)(1+\sin t)$ for $0 \le t \le \pi/2$. Now

$$\begin{aligned}(1+\cos t)(1+\sin t) &= 1 + \sin t + \cos t + \sin t \cos t \\ &= 1 + \sqrt{2}\sin(t+\pi/4) + \tfrac{1}{2}\sin(2t).\end{aligned}$$

The second and third terms increase from $t = 0$ to $\pi/4$ then decrease from $t = \pi/4$ to $\pi/2$, so the maximum occurs at $t = \pi/4$. Since the values at $t = 0$ and $t = \pi/2$ are equal, the minimum occurs here. This gives

$$4(1+\cos\pi/4)(1+\sin\pi/4) = 6 + 4\sqrt{(2)} \approx 11.6569$$

as the maximum value and

$$4(1+\cos 0)(1+\sin 0) = 8$$

as the minimum value.

OC522. Find the largest natural number n such that the sum

$$\lfloor\sqrt{1}\rfloor + \lfloor\sqrt{2}\rfloor + \cdots + \lfloor\sqrt{n}\rfloor$$

is a prime number.

Originally from 2018 Czech-Slovakia Math Olympiad, 4th Problem, Category A, First Round.

We received 13 *submissions, of which* 10 *were correct and complete. We present the solution by the Missouri State University Problem Solving Group.*

Let

$$f(n) = \lfloor\sqrt{1}\rfloor + \lfloor\sqrt{2}\rfloor + \cdots + \lfloor\sqrt{n}\rfloor.$$

Note that $f(47) = 197$, which is prime. We claim that this is the largest n such that $f(n)$ is prime.

If $k^2 \le n \le (k+1)^2 - 1$, then

$$\begin{aligned}f(n) = \left(\sum_{i=1}^{k-1}\sum_{j=i^2}^{(i+1)^2-1}\lfloor\sqrt{j}\rfloor\right) + \sum_{j=k^2}^{n}\lfloor\sqrt{j}\rfloor &= \left(\sum_{i=1}^{k-1}\sum_{j=i^2}^{(i+1)^2-1} i\right) + \sum_{j=k^2}^{n} k \\ &= \left(\sum_{i=1}^{k-1}(2i+1)i\right) + (n+1-k^2)k \\ &= \frac{k(k-1)(4k+1)}{6} + (n+1-k^2)k.\end{aligned}$$

Now $f(48) = 203 = 7 \cdot 29$ is not prime.

If $n \geq 49$, then $k > 6$. Let $m = \dfrac{k(k-1)(4k+1)}{6} \in \mathbb{Z}$. We claim $\gcd(m,k) > 1$. If not, then $k(k-1)(4k+1) = 6m$ and $\gcd(m,k) = 1$ would imply that k divides 6, which contradicts the fact that $k > 6$. If $d = \gcd(m,k)$, then d is a nontrivial factor of $f(n)$, since $d > 1$ and $d \leq k < m \leq f(n)$. Therefore $f(n)$ is not prime.

Editor's Comment. UCLan Cyprus Problem Solving Group observed that this is sequence A022554 in OEIS. In the comments for the sequence it is mentioned that "It seems that 197 is the largest prime in this sequence....". So, here it has been confirmed that this is indeed the case.

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