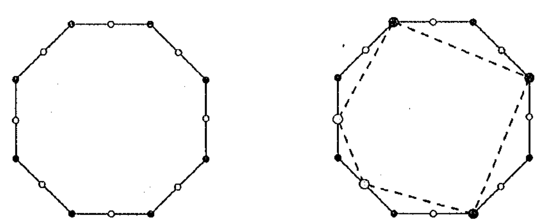
MA64. A regular octagon is shown in the first diagram below, with the vertices and midpoints of the sides marked.

An "inner polygon" is a polygon formed by traversing the octagon in a clockwise manner, selecting some of the marked points as you go, ensuring that each side of the original octagon contains exactly one selected point. Then each selected point is connected to the next with a line segment, and the last is connected to the first to complete the inner polygon.

An example of an inner polygon is shown in the second diagram.

How many inner polygons does the regular octagon have?



Originally problem 13 of the 2017 Mathworks Math Contest (with modified wording).

We received three submissions, out of which two were correct and complete. Both are presented below.

Solution 2, by the Missouri State University Problem Solving Group, slightly edited.

We claim that for any regular n-gon with $n \geq 5$, the number of inner polygons is $F_{n+1} + F_{n-1}$, where F_k is the k-th Fibonacci number. In particular, when $n = 8$, we have $F_9 + F_7 = 34 + 13 = 47$ inner polygons. [*Ed: The formula also holds for* $n < 5$ *if we allow degenerate polygons*]. Note that $F_{n+1} + F_{n-1}$ is also known as the n-th Lucas number L_n.

We first show that the number of sequences of 0's and 1's of length n having no two consecutive 1's is F_{n+2}. We call such a sequence *nice.* Let A_n denote the number of nice sequences of length n. Then $A_1 = 2 = F_3$ and $A_2 = 3 = F_4$. There is a bijection between nice sequences of length n starting with 0 and nice sequences of length $n - 1$ obtained by deleting the leading 0. There is also a bijection between nice sequences of length n starting with 1 and nice sequences of length $n - 2$, obtained by deleting the leading two terms, which have to be 1 and 0 by definition. Therefore $A_n = A_{n-1} + A_{n-2}$. Since the Fibonacci sequence obeys the same recursion with shifted initial terms, the result follows.

We next note that there is a bijection between inner polygons and nice sequences that do not both begin and end with a 1. Fix a vertex of the original polygon. Beginning at that vertex and circling through the other vertices of the original polygon, write a 1 if the vertex is a vertex of the inner polygon and a 0 if it is not. By definition, no two 1's can be consecutive, nor can both the first and the last terms be 1's. Conversely, given a sequence, we construct an inner polygon by taking as vertices all the vertices of the original polygon that correspond to a 1 in the sequence and all the midpoints of sides of the original polygon that do not contain one of those vertices.

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Finally, we show that the number of such sequences is as claimed. If a sequence begins with a 0, we can append a nice sequence of length $n-1$, of which there are F_{n+1}. If the sequence begins with a 1, then the second and last term must both be 0. This leaves a nice sequence of length $n-3$, of which there are F_{n-1}. Therefore the total number of inner polygons is $F_{n+1}+F_{n-1}$.