OC606. Determine the number of triples of positive integers (a, b, c) such that

$$a + ab + abc + ac + c = 2017.$$

Originally from 2018 Czech-Slovakia Mathematics Olympiad, 4th Problem, Category B.

Solution 2, by Missouri State University Problem Solving Group.

Solving for c gives

$$c = \frac{2017 - a - ab}{ab + a + 1} = \frac{2018}{ab + a + 1} - 1.$$

Therefore $ab+a+1$ must be a divisor of 2018, i.e., $ab+a+1 = 1, 2, 1009$, or 2018. Hence

$$a(b + 1) = 0, 1, 1008, \text{ or } 2017.$$

The first two possibilities cannot occur if $a, b > 0$. If $ab + a = 2017$, then $c = 0$. Thus, the only possibility is $a(b + 1) = 1008$, which gives $c = 1$. Now $b + 1$ can be any divisor of 1008, except 1 and there are no restrictions on a. Since 1008 has 30 positive divisors, this gives 29 solutions.

A similar argument shows that, more generally, if n is a positive integer and $f(n)$ denotes the number of solutions to

$$a + ab + abc + ac + c = n,$$

then

$$f(n) = \sum_{\substack{d|(n+1) \\ d \neq 1, 2, n+1}} (\tau(d - 1) - 1),$$

where $\tau(m)$ is the number of positive divisors of m.

For example, if 2017 is replaced by 2023, we have 49 solutions.