Point and Oval Incidences in a Projective Plane

Yunus Syed [1]

University of Illinois at Chicago

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Overview

- Projective Planes, Arcs, and Ovals
- Oval Counting
- Incidence Bounds

Incidence Systems

Before working with abstract projective planes, one must understand incidence systems.

- An incidence system, $\mathscr{S}$, is a triple $(\mathscr{P}, \mathscr{L}, \mathscr{I})$ where $\mathscr{P}$, $\mathscr{L}$ are sets and $\mathscr{I} \subset \mathscr{P} \times \mathscr{L}$.
- We think of $\mathscr{P}$ as the set of points, $\mathscr{L}$ as the set of lines, and $\mathscr{I}$ as the set of incidences between points and lines. Formally speaking, if $(p, \ell) \in \mathscr{I}$, we say the point p is incident to the line ℓ.
- $\mathscr{S}$ is a finite system if both $\mathscr{P}$ and $\mathscr{L}$ are finite.

Abstract Projective Planes

An incidence system, $\mathscr{S}$, is a projective plane if it satisfies three axioms:

1. Any two distinct points are incident with exactly one line.
2. Any two distinct lines are incident with exactly one point (Every pair of lines has a unique intersection).
3. There exist four points such that no three are incident with the same line, i.e a quadrilateral.

Finite Projective Planes

A projective plane, $\mathscr{S}$ is a finite projective plane of order n whenever:

1. $|\mathscr{P}| = |\mathscr{L}| = n^2 + n + 1$.
2. Each $p \in \mathscr{P}$ is incident to exactly $n + 1$ lines and each $\ell \in \mathscr{L}$ contains exactly $n + 1$ points.

The easiest to draw examples are the projective planes of orders 2 and 3. From now on, we work only in the realm of finite projective spaces of order n.

Arcs and Ovals

- An arc is a set of points such that no three are colinear. We can think of quadrilaterals as arcs with four elements.
- An oval is an arc with $n + 1$ elements and a hyperoval is an arc with $n + 2$ elements.
- Let $\mathscr{O}$ denote the set of all ovals in our projective plane.

What's Next?

1. A nice bound on ovals that pass through a given set of points
2. Ongoing work in reducing this bound
3. Using the bounds from Part 1 to determine bounds on point and oval incidences
4. Counting arcs

Our Little Lemma

Lemma 1.1: Let $S \subset \mathscr{P}$ be non-empty and let $\mathscr{O}_S = \{O \in \mathscr{O} : S \subset O\}$. Then, if $|S| = k$, we have $|\mathscr{O}_S| \leq (n - k + 2)n^{n-k+1}$.
The main idea of the proof is fairly simple but the formal write up is $\frac{3}{4}$ of a page.

Smaller and smaller

We can reduce the bounds found in Lemma 1.1 by using not so nice methods. For starters, observe that we have over counted by a lot.

Straightforward Incidence Bounds

We denote the set of all incidences between $\mathscr{P}$ and $\mathscr{O}$ as $I(\mathscr{P}, \mathscr{O})$. Using this notation, Lemma 1.1 gives rise to two simple incidence bounds.

1. If $|\mathscr{P}| = 1$, then $|I(\mathscr{P}, \mathscr{O})| \leq (n+1)n^n$. This follows from Lemma 1.1 by letting $k = 1$.
2. Similarly, if $|\mathscr{P}| = 2$, then $|I(\mathscr{P}, \mathscr{O})| \leq 2(n+1)n^n$. The proof of this result can be visualized by a venn diagram.
3. In the situation when $|\mathscr{P}| \geq 3$, there is no guarantee that all the points of $\mathscr{P}$ can be contained on an oval. However, we need not worry.

An Exact Incidence Count

Theorem 2.3: If $|\mathscr{P}| = k$, then $|I(\mathscr{P}, \mathscr{O})| = \sum_{i=1}^{k} |\mathscr{O}_i|$.

- The main idea of the proof is similar to the cases when $k = 2$ and $k = 3$.
- In the general case, for every $Q \subset \mathscr{P}$ such that $|Q| > 1$, the coefficient of $|\mathscr{O}_Q|$ in $|I(\mathscr{P}, \mathscr{O})|$ is 0 by a nice combinatorial identity.

An Interesting Relation

We immediately have one incidence bound and a bound on the number of ovals.

- **Corollary 2.4:** If $|\mathscr{P}| = k$, then $|I(\mathscr{P}, \mathscr{O})| \leq k(n+1)n^n$.
- **Corollary 2.5:** $|\mathscr{O}| \leq (n^2 + n + 1)n^n$.

The second inequality comes from letting $\mathscr{P}$ be all points in the projective plane and observing that
$(n+1)\,|\mathscr{O}| = |I(\mathscr{P}, \mathscr{O})| = (n^2+n+1)\,|\mathscr{O}_1| \leq (n^2+n+1)(n+1)n^n$.
The identity $(n+1)\,|\mathscr{O}| = (n^2 + n + 1)\,|\mathscr{O}_1|$ gives us one way to count ovals.

Direct Arc Counting

The second way is by constructively counting arcs point-wise.

- Choose your first point p_1. There are no restrictions so the number of options is $n^2 + n + 1$.
- Choosing a second point p_2 is almost as easy. There are $n^2 + n$ options.
- Things become interesting with the third point p_3 since p_3 can not be contained on the line determined by p_1 and p_2.
- Likewise, the point p_i can not be on any of the lines determined by the points $p_1, p_2, \cdots p_{i-1}$.

A Weak Lemma + Conjecture

Let $\mathscr{A}_k$ denote the set of all k-arcs in a projective plane of order n.

- **Lemma:** $k < 7$, $|\mathscr{A}_k| = \frac{1}{(k)!}\prod_{i=1}^{k}((n^2+n+1)-(\binom{i-1}{2}(n-1)+(i-1)-(\binom{i-3}{2})(i-2)))$.
- When $k \geq 7$, we run into issues. As a result, our counting needs to be more careful.
- **Conjecture:** For k such that $n > \frac{k^2-3k}{4} + \frac{\sqrt{k^4-14k^3+73k^2+232k+160}}{4}$, $|\mathscr{A}_k| \geq \frac{1}{(k)!}\prod_{i=1}^{k}((n^2+n+1)-(\binom{i-1}{2}(n-1)+(i-1)-(\binom{i-3}{2})(i-2)))$

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- Solve some more problems and win the Fields Medal?